MATH 1A − QUIZ 8 − SOLUTIONS

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(1) *(4 points; 1 point each)* Let $f(x) = x^5 - 5x^4$

(a) Find the intervals of increase and decrease

$$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$$

Now $f'(x) = 0 \Leftrightarrow x = 0$ or $x = 4$.

Now drawing a sign table (Note that you did not have to calculate $f(0)$ and $f(4)$), we get:
1A/Math 1A - Fall 2013/Quizzes/Quiz8table1.png

x	- oo	0		4	oo
$5x^3$	−	0	+		+
x - 4	−		−	0	+
f'(x)	+	0	−	0	+
f(x)	↗	0	↘	-256	↗

From this table, we can see that:
- f is increasing on $(-\infty, 0) \cup (0, \infty)$
- f is decreasing on $(0, 4)$

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(b) Find the local maxima and minima.

Using the table above and the first derivative test, we get that $\boxed{f \text{ has a local maximum at } 0}$ (f' changes sign from positive to negative there)

And $\boxed{f \text{ has a local minimum at } 4}$ (f' changes sign from negative to positive there).

(c) Find the intervals of concavity and inflection points.

$$f''(x) = 20x^3 - 60x^2 = 20x^2(x-3)$$

Then $f''(x) = 0 \Leftrightarrow x = 0$ or $x = 3$. Now using the following sign table, we can determine the concavity and inflection points of f:

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x	- oo	0		3	oo
20x²	+	0	+		+
x - 3	−		−	0	+
(x)	−	0	−	0	+
f(x)	C.D.		C.D.	I.P.	C.U.

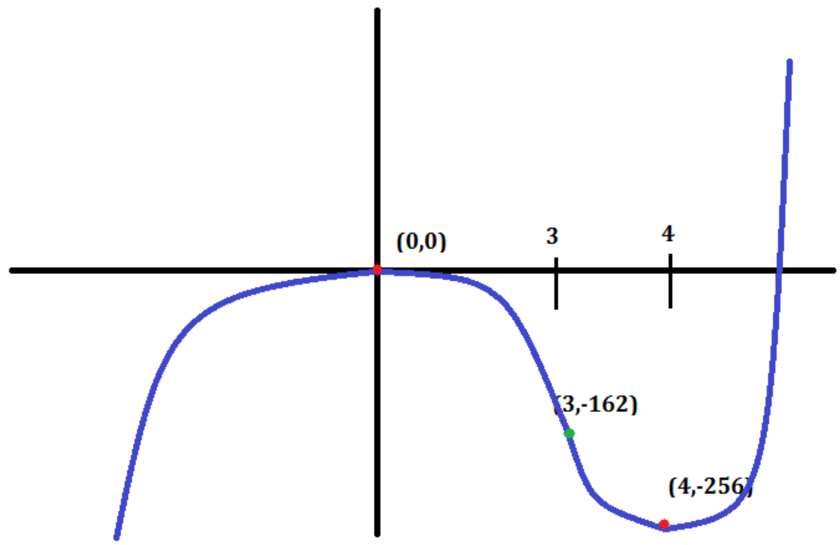
From this table, we can see that:
- f is concave up on $(3, \infty)$
- f is concave down on $(-\infty, 0) \cup (0, 3)$ (or $(-\infty, 3)$ if you'd like)

In particular, f changes concavity at 3, so f has an inflection point at $\boxed{x = 3}$. (Note that f does **NOT** have an inflection point at 0 because f **DOESN'T** change concavity there!)

(d) Sketch the graph of f

Again, note that the actual values of -162 and -256 were not required.

1A/Math 1A - Fall 2013/Quizzes/Quiz8graph.png



(2) *(3 points; 1.5 points each)* Evaluate the following limits:

(a) $\lim_{x\to 0^+} x\left(\ln(x)\right)^2$

By using l'Hopital's rule twice (both times having the form $\frac{\infty}{\infty}$, we get:

$$\begin{aligned}\lim_{x\to 0^+}\left(x\left(\ln(x)\right)^2\right) &= \lim_{x\to 0^+}\frac{\left(\ln(x)\right)^2}{\frac{1}{x}}\\ &\overset{H}{=}\lim_{x\to 0^+}\frac{2\frac{\ln(x)}{x}}{-\frac{1}{x^2}}\\ &=\lim_{x\to 0^+} -2x\ln(x)\\ &=\lim_{x\to 0^+}\frac{-2\ln(x)}{\frac{1}{x}}\\ &\overset{H}{=}\lim_{x\to 0^+}\frac{\frac{-2}{x}}{-\frac{1}{x^2}}\\ &=\lim_{x\to 0^+} 2x\\ &=0\end{aligned}$$

(b) $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$ (here a and b are constants)

Notice that this is of the form 1^∞, hence:

(1) Let $y=\left(1+\frac{a}{x}\right)^{bx}$

(2) $\ln(y)=bx\ln\left(1+\frac{a}{x}\right)$

$$\begin{aligned}\lim_{x\to\infty}\ln(y) &= \lim_{x\to\infty} bx\ln\left(1+\frac{a}{x}\right)\\ &=\lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{bx}}\\ &\overset{H}{=}\lim_{x\to\infty}\frac{\frac{1}{1+\frac{a}{x}}\times\frac{-a}{x^2}}{-\frac{1}{b(x^2)}}\\ &=\lim_{x\to\infty} ab\left(\frac{1}{1+\frac{a}{x}}\right)\\ &=ab\left(\frac{1}{1+0}\right)\\ &=ab\end{aligned}$$

(3) Hence $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=e^{ab}$

(3) *(3 points)* We say that x is a **fixed point** of f if $f(x) = x$

(for example, -1 is a fixed point of x^3 because $(-1)^3 = -1$)

Show that if $f'(x) \neq 1$ for all x, then f has at most one fixed point.

Suppose that f has at least 2 fixed points a and b. Then $f(a) = a$ and $f(b) = b$ (by definition of fixed points).

Now, since f is differentiable on $[a, b]$, by the **Mean Value Theorem** on $[a, b]$, we get that for some c in (a, b):

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

But $\frac{f(b)-f(a)}{b-a} = \frac{b-a}{b-a} = 1$, so we get:

$$f'(c) = 1$$

But this contradicts the fact that $f'(x) \neq 1$ for all $x \Longrightarrow\Longleftarrow$

Hence f has at most one fixed point.